Noise-induced transport of Brownian particles with consideration for their mass

P. S. Landa
Department of Physics, Lomonosov Moscow State University, 119899 Moscow, Russia
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The noise-induced transport of Brownian particles with regard to their mass is considered. The results of approximate analytical calculations for the averaged particle flux in periodic ratchetlike potentials are presented. It is shown that with increase in mass the reversal of flux is possible. An analogy between noise-induced transport and well known in mechanics vibrational transport is discussed. [S1063-651X(98)10607-4]

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INTRODUCTION

In recent years phenomena of noise-induced transport of Brownian particles has attracted the considerable interest of many scientists, for the most part in the context of different biological and chemical problems (see, for example, Refs. [1–6]). A physical experiment demonstrating the possibility of such transport in a ratchetlike potential field is described in Ref. [7].

Systems in which noise-induced transport occurs are often called stochastic ratchets in analogy to the mechanical device ''ratchet-and-pawl'' described and considered by Feynman [8]. Feynman showed that in the case of thermodynamical equilibrium the ratchet on average is at rest as it must because of the second law of thermodynamics.

It should be noted that similar phenomena were also discussed prior to Feynman lectures [9–15]. In Refs. [11,12] it was shown that in the simplest electrical rectifier consisting of condenser and diode the condenser can be charged without an external source, only at the sacrifice of thermal fluctuations. This paradoxical result cast some doubt on the feasibility of the second thermodynamics law as applied to the phenomenon considered [14]. As far back as 1950, Brillouin [10] showed, considering diode as a nonlinear resistor, that for the feasibility of the second thermodynamics law a shift of the voltage-current characteristic of the nonlinear resistor must be taken into account. Stratonovich [15] established, on a certain model of the diode, that such a shift does occur and calculated it. With this shift the mean value of the voltage drop across the condenser and the mean current in the circuit vanish.

We note that in works concerning noise-induced transport authors allege that a necessary condition for existence of a directional motion of Brownian particles is the presence of a spatioperiodic potential with a certain asymmetry. According to them the periodicity is required to permanently extract work from applied fluctuations, thus transforming noise into directed motion. The examples considered in the first section demonstrate that this condition is not necessary.

In the last few years much attention has been concentrated on the problem associated with the separation of particles of different mass or size. In this connection studies of different models giving flux reversals as the system parameters change [16–19] are very important. Below it is shown that under certain conditions flux reversal is possible with increasing particle mass.

As a rule, the consideration of noise-induced transport is restricted to the so-called overdamped case when the motion of a Brownian particle is described by a first order differential equation of the form

$$\dot{x} = -f(x) + \zeta(x,t) + \xi(t), \tag{1}$$

where $f(x)$ is a periodic function of x possessing a certain asymmetry, $\zeta(x,t)$ is a random process with zero mean value, and $\xi(t)$ is white noise imitating thermal fluctuations. The process $\zeta(x,t)$ can be either given or described by other equations.

It is usual to distinguish two types of ratchet devices [4–6,20,21]: (1) $\zeta(x,t)$ is independent of x (fluctuating force) and (2) $\zeta(x,t)$ depends on x (fluctuating barrier). In its turn, the latter can be also divided into two classes: (a) $\zeta(x,t) = f(x)\chi(t)$ [4] and (b) $\zeta(x,t)$ is a random function of t and x [21].

We restrict ourselves to the first type of ratchet devices, but take into account particle mass. In addition, we show that there is a certain analogy between noise-induced and vibrational transport that is well known in mechanical engineering [22].

I. AN ELECTRICAL RECTIFIER AND VIBRATIONAL TRANSPORT

First of all let us consider an electrical rectifier shown in Fig. 1. Taking into account the shift of the diode current-voltage characteristic we can write the following equation for the voltage drop across the diode V:

$$\dot{V} = F(V - V_0) - V/\tau + \xi(t), \tag{2}$$

where $CF(V - V_0)$ is the current flowing through the diode, $\tau = RC$ is the relaxation time, and $\xi(t)$ is white noise of intensity K.

For simplicity we set $F(v)$ in the form

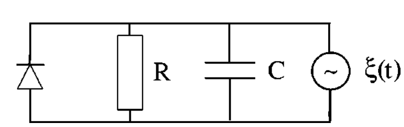

FIG. 1. Schematic image of an electrical rectifier.

$$F(v)=\begin{cases}-a_1 v & \text{for } v>0,\\ -a_2 v & \text{for } v<0.\end{cases}$$

The value of V_0 can be calculated by using the technique suggested by Stratonovich [23]. In so doing we find

$$V_0=\sqrt{\frac{K_0\tau(1+a_1\tau)(1+a_2\tau)}{\tau}}\,\frac{(a_1-a_2)\tau}{\sqrt{1+a_1\tau}+\sqrt{1+a_2\tau}},\qquad(3)$$

where K_0 is the intensity of thermal fluctuations.

Solving the corresponding Fokker-Planck equation we obtain the following expression for the stationary probability density:

$$w(V)=C\begin{cases}\exp\left(-\dfrac{(1+a_1\tau)(V-V_0)^2}{K\tau}\right) & \text{for } V>V_0\\ \exp\left(-\dfrac{(1+a_2\tau)(V-V_0)^2}{K}\tau\right) & \text{for } V<V_0,\end{cases}\qquad(4)$$

where

$$C=\frac{2\sqrt{(1+a_1\tau)(1+a_2\tau)}}{\pi K\tau(\sqrt{1+a_1\tau}+\sqrt{1+a_1\tau})}$$

is the normalization constant.

If the intensity of noise $\xi(t)$ is greater than or equal to K_0 then

$$\langle V\rangle=\sqrt{\frac{\tau}{\pi}}\,(\sqrt{K}-\sqrt{K_0})(a_2-a_1)\tau\,\frac{\sqrt{(1+a_1\tau)(1+a_2\tau)}}{\sqrt{1+a_1\tau}+\sqrt{1+a_2\tau}}.\qquad(5)$$

It follows that $\langle V\rangle$ is equal to zero for $K=K_0$ and not equal to zero for $K>K_0$. The sign of $\langle V\rangle$ is determined by the sign of the difference a_2-a_1. So, as would be expected, for $K>K_0$ and $a_1\neq a_2$ we obtain the rectification of fluctuations, i.e., directed motion of electrons caused by noise. It is evident that this phenomenon is similar to a stochastic ratchet.

Let us now consider the simplest example of vibrational transport and show that it is akin to electron transport in an electrical rectifier. Let a body of mass m lies on a horizontal plane vibrating in the direction of an axis x. We assume that a force of dry friction between the body and plane has different values for $\dot{x}>0$ and $\dot{x}<0$. This is possible if the surface of the plane is rough. Then we can write the following equation for the body motion:

$$\dot{y}=-f(y)+F(t),\qquad(6)$$

where $y=\dot{x}$, $F(t)$ is proportional to the inertial force due to vibration of the plane, and

$$f(y)=\begin{cases}a_1 & \text{for } y>0\\ -a_2 & \text{for } y<0,\end{cases}\qquad(7)$$

$a_{1,2}$ are the friction factors.

In mechanics it is usual to consider harmonically vibrating plane, i.e., $F(t)=B\sin\omega t$. In this case the availability or lack of transport are determined by the value of B and the difference between a_1 and a_2. If $B<\min a_{1,2}$ then the body, being for $t=0$ at rest, remains immobile for all t. In the case of $a_1<B<a_2$ the body moves towards the right during the time lapse between t_1+nT and t_2+nT, where $t_1=(1/\omega)\arcsin(a_1/B)$, t_2 is determined by the equation $(B/\omega)(\cos\omega t_1-\cos\omega t_2)=a_1(t_2-t_1)$, n is an integer and $T=2\pi/\omega$; during the remainder of time the body is at rest. It is evident that $\overline{y}=1/T\int_0^T y(t)\,dt>0$. So, the body moves on average, though no constant forces act upon it; in the process the motion occurs in the direction of less resistance offered by the friction force. In the case of the most interest, that $B>\max a_{1,2}$, the body moves towards both the right and the left, but in the average it moves in the direction of the less resistance as before. Let us consider this case in more detail in the time interval $0\leq t\leq T$. A solution of Eq. (6), in view of Eq. (7), is

$$y(t)=\begin{cases}y(0)+a_2 t+\dfrac{B}{\omega}(1-\cos\omega t) & \text{for } 0\leq t\leq t_1\\ -a_1(t-t_1)+\dfrac{B}{\omega}(\cos\omega t_1-\cos\omega t) & \text{for } t_1\leq t\leq t_2\\ a_2(t-t_2)+\dfrac{B}{\omega}(\cos\omega t_2-\cos\omega t) & \text{for } t_2\leq t\leq T,\end{cases}\qquad(8)$$

where $y(0)$, t_1, and t_2 are determined by the following equations:

$$y(0)+a_2 t_1+\frac{B}{\omega}(1-\cos\omega t_1)=0,$$

$$-a_1(t_2-t_1)+\frac{B}{\omega}(\cos\omega t_1-\cos\omega t_2)=0,\qquad(9)$$

$$a_2(T-t_2)+\frac{B}{\omega}(\cos\omega t_2-1)-y(0)=0.$$

An example of the plot of $y(t)$ is given in Fig. 2.

It follows from Eqs. (8),(9) that

$$\overline{y}=\frac{1}{\omega}\sqrt{B^2-\frac{\pi^2 a_1^2 a_2^2}{(a_1+a_2)^2}\left(\sin\frac{\pi a_2}{a_1+a_2}\right)^{-2}}\cos\frac{\pi a_1}{a_1+a_2}.\qquad(10)$$

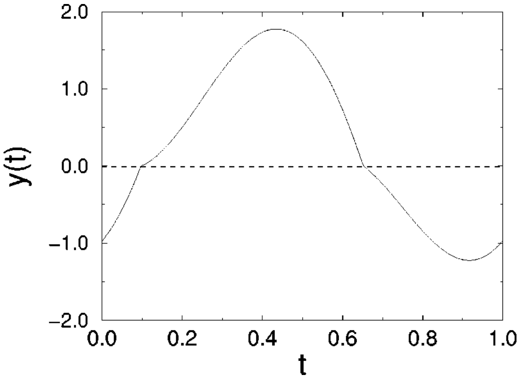

FIG. 2. The plot of $y(t)$ for $a_1=5.024$, $a_2=2\pi$, $B/\omega=2$, and $\omega=2\pi$ for values of parameters $t_1=0.0971137$, $t_2=0.65267$, $y(0)=-0.970434$, $\overline{y}\approx 0.2461$, $B_0/\omega\approx 1.4175$.

It is easily seen that this expression is valid for $B\geqslant B_0$, where

$$B_0=\frac{\pi a_1 a_2}{a_1+a_2}\left(\sin\frac{\pi a_2}{a_1+a_2}\right)^{-1}.$$

We can see from Eq. (10) that, as one would expect, $\overline{y}$ is equal to zero for $a_1=a_2$, positive for $a_1<a_2$, and negative for $a_1>a_2$.

Let us consider further the case of random vibration. We put $F(t)=\xi(t)$, where $\xi(t)$ is sufficiently wide-band noise of intensity K with zero mean value. In this case we can use the Fokker-Planck equation associated with the Langevin equation (6). The stationary solution of this equation satisfying the condition for the probability flux to be zero is

$$w(y)=C\exp\left(\frac{2}{K}\int_0^y f(y')dy'\right), \tag{11}$$

where the constant C is determined from the normalization condition. It is

$$C=\left[\int_{-\infty}^{\infty}\exp\left(\frac{2}{K}\int_0^y f(y')dy'\right)dy\right]^{-1}. \tag{12}$$

Using the expressions (11),(12) we can find the mean value of y:

$$\langle\varphi(y)\rangle=\int_{-\infty}^{\infty} y\exp\left(\frac{2}{K}\int_0^y f(y')dy'\right) \times dy\left[\int_{-\infty}^{\infty}\exp\left(\frac{2}{K}\int_0^y f(y')dy'\right)\,dy\right]^{-1}. \tag{13}$$

If $f(y)$ is described by the expression (7) then

$$\langle y\rangle=\frac{K}{2a_1a_2}\,(a_2-a_1). \tag{14}$$

So, in this case we obtain the same result as for harmonic vibration: for $a_1=a_2$ the body is on average at rest, whereas for $a_1\neq a_2$ the body moves on average in the direction of the less resistance. This result is similar to the rectification of fluctuations. However, there is a dissimilarity from the case of harmonic vibration: in the case of harmonic vibration the effect is of threshold character, whereas in the case of random vibration the transport can occur for the fluctuation intensity as small as is wished.

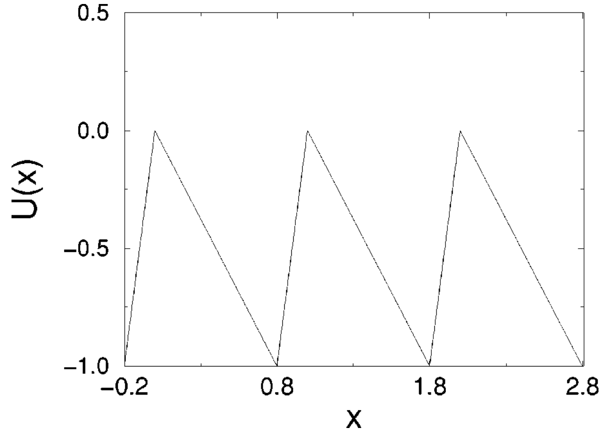

FIG. 3. An example of the sawtooth potential.

II. TRANSPORT OF A MASSIVE BROWNIAN PARTICLE IN A VISCOUS MEDIUM WITH SAWTOOTH POTENTIAL UNDER THE ACTION OF REGULAR AND RANDOM FORCES

Let us consider the motion of a Brownian particle in a viscous medium described by the following equation:

$$\mu\ddot{x}+\dot{x}+f(x)=\zeta(t)+\xi(t), \tag{15}$$

where $\mu=m/\beta$, m is the particle mass, β is the viscous friction factor, $f(x)$ is a periodic function of x, $\zeta(t)$ is a function of time which can be both regular and random, and $\xi(t)$ is white noise of intensity K imitating thermal fluctuations. For definiteness, we set

$$f(x)=\begin{cases} a_1 & \text{for } nL<x<nL+x_1 \\ -a_2 & \text{for } nL-x_2<x<nL, \end{cases} \tag{16}$$

where $n=0,\pm1,\pm2,\ldots$, $L=x_1+x_2$ is the period of the function $f(x)$. We note that such a form of the function $f(x)$ corresponds to the sawtooth potential $U(x)$ shown in Fig. 3.

The function $f(x)$ can be expanded into the Fourier series

$$f(x)=\sum_{n=1}^{\infty}\frac{1}{n\pi}\left((a_1+a_2)\sin\frac{2\pi nx}{L}-a_1\sin\frac{2\pi n(x-x_1)}{L} -a_2\sin\frac{2\pi n(x+x_2)}{L}\right). \tag{17}$$

We note that for any finite number of the series terms $f(x)$ is a differentiable function. The shape of the function $f(x)$ and the potential $U(x)$ in the approximation of four terms of the series (17) is shown in Fig. 4.

The problem is to calculate the particle velocity $\dot{x}$ averaged over both statistical ensemble and time.

This problem can be solved analytically only in the case that μ is sufficiently small. As will be seen from the subsequent results, the condition of smallness is

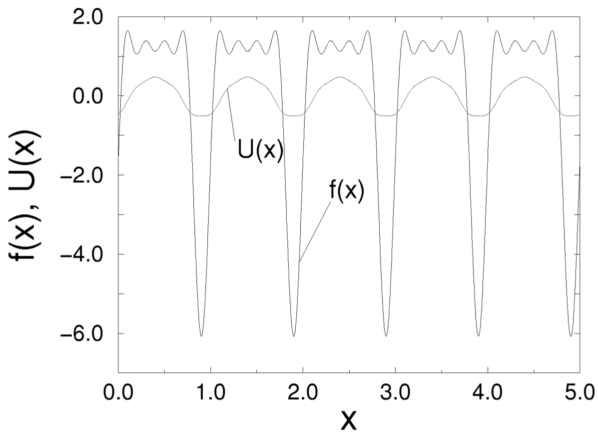

FIG. 4. The shape of the function $f(x)$ and the potential $U(x)$ in the approximation of four terms of the series (17).

$$\mu \frac{a_1 a_2}{K} \ll 1. \tag{18}$$

At the condition (18) we can use the approximate one-dimensional Fokker-Planck equation for the probability density of the variable x derived by Stratonovich [24]:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[[1+\mu f'(x)] \left([f(x)-\zeta(t)] w(x,t) + \frac{K}{2} \frac{\partial}{\partial x} \right) \right], \tag{19}$$

where $f'(x)=df(x)/dx$. Although the term $f'(x)$ enters into Eq. (19), it disappears in the expression for $w(x,\zeta)$; therefore we can take $f(x)$ in the form of Eq. (16). For $\mu=0$ Eq. (19) is the exact Fokker-Planck equation corresponding to the Langevin equation (15).

Because $f(x)$ is a periodic function of x, the probability density $w(x,\zeta)$ is also a periodic function of x. The latter enables Eq. (19) to be solved only on the interval from $-x_2$ to x_1. If the function $\zeta(t)$ varies sufficiently slowly, we can use so-called quasistationary approximation for solving Eq. (19), i.e., neglect the term $\partial w/\partial t$. In this approximation we obtain from Eq. (19) the following equation for $w(x,\zeta)$:

$$G(t) = -[1+\mu f'(x)] \left([f(x)-\zeta(t)] w + \frac{K}{2} \frac{\partial w}{\partial x} \right), \tag{20}$$

where $G(t)$ is the probability flux in the direction of the axis x in the instant t.

Let us show that the particle velocity $\dot{x}$ averaged over both statistical ensembles is proportional to the probability flux $G(t)$:

$$\langle \dot{x} \rangle = G(t) L, \tag{21}$$

where the symbol $\langle\,\rangle$ denotes averaging over ensemble, $L = x_1 + x_2 = U(a_1+a_2)/a_1 a_2$ is the period of the function $f(x)$.

First of all let us note that Eq. (19) is equivalent to the following Langevin equation [24]:

$$\dot{x} = [1+\mu f'(x)][\zeta(t) - f(x)] + \mu \frac{K}{2} f''(x) + \sqrt{[1+\mu f'(x)]}\,\xi(t). \tag{22}$$

Averaging Eq. (22) with regard to the correlation between x and $\xi(t)$ [24], we obtain

$$\langle \dot{x} \rangle = \int_{-x_2}^{x_1} \left([1+\mu f'(x)][\zeta(t)-f(x)] + \mu \frac{K}{2} f''(x) \right) w(x,t)\, dx. \tag{23}$$

Taking into account Eq. (20), we can rewrite the integrand in (23) as

$$\left([1+\mu f'(x)][\zeta(t)-f(x)] + \mu \frac{K}{2} f''(x) \right) w(x,t) = G(t) + \frac{K}{2} \frac{\partial}{\partial x} \{[1+\mu f'(x)] w(x,t)\}.$$

Substituting this expression in (23) and using the periodicity conditions for the functions $w(x,t)$ and $f(x)$ we obtain the formula (21). Averaging further (21) over time we find

$$\overline{\langle \dot{x} \rangle} = \overline{G(t)} L, \tag{24}$$

where

$$\overline{G} = \lim_{T\to\infty} \int_0^T G(t)\, dt.$$

Thus, the transport of particles is determined by the mean value of the probability flux $G(t)$.

A solution of Eq. (20) is

$$w(x,t) = 2\mu \frac{G(t) f(x)}{K} + \exp\left(-\frac{2[U(x)-\zeta x]}{K} \right) \times \left[C(t) - \frac{2G(t)}{K} \int_0^x \left(1 + 2\mu \frac{f(x')[f(x')-\zeta]}{K} \right) \times \exp\left(\frac{2[U(x')-\zeta x']}{K} \right) dx' \right], \tag{25}$$

where $U(x) = \int_0^x f(x')\, dx'$ is the potential.

For $f(x)$ described by the expression (16) we obtain

$$w(x,t) = \begin{cases} \dfrac{G(t)}{\zeta - a_1} + \left(C(t) - \dfrac{G(t)}{\zeta - a_1} + 2\mu \dfrac{G(t) a_1}{K} \right) \exp\left(\dfrac{2(\zeta - a_1)x}{K} \right) & \text{for } 0 < x < x_1 \\ \dfrac{G(t)}{\zeta + a_2} + \left(C(t) - \dfrac{G(t)}{\zeta + a_2} - 2\mu \dfrac{G(t) a_2}{K} \right) \exp\left(\dfrac{2(\zeta + a_2)x}{K} \right) & \text{for } -x_2 < x < 0, \end{cases} \tag{26}$$

where $C(t)$ is an arbitrary function of t.

From the periodicity condition of the function $w(x,t)$ we find the relationship between $C(t)$ and $G(t)$:

$$C(t)\left[\exp\left(\frac{2U_0\zeta}{Ka_1}\right)-\exp\left(-\frac{2U_0\zeta}{Ka_2}\right)\right]=G(t)\left[\left(\frac{1}{\zeta-a_1}-\frac{2\mu a_1}{K}\right)\exp\left(\frac{2U_0\zeta}{Ka_1}\right)-\left(\frac{1}{\zeta+a_2}+\frac{2\mu a_2}{K}\right)\right.$$
$$\left.\times\exp\left(-\frac{2U_0\zeta}{Ka_2}\right)-\frac{a_1+a_2}{(\zeta-a_1)(\zeta+a_2)}\exp\left(\frac{2U_0}{K}\right)\right]. \tag{27}$$

Taking into account Eq. (27), from the normalization condition we determine G:

$$G(t)=U_0\left[\exp\left(\frac{2U_0\zeta}{Ka_1}\right)-\exp\left(-\frac{2U_0\zeta}{Ka_2}\right)\right]\left(\frac{1}{a_1(\zeta-a_1)}+\frac{1}{a_2(\zeta+a_2)}\right)+\frac{K(a_1+a_2)^2e^{-2U_0/K}}{2(\zeta-a_1)^2(\zeta+a_2)^2}\left[\exp\left(\frac{2U_0}{K}\right)-\exp\left(-\frac{2U_0\zeta}{Ka_2}\right)\right]$$
$$\times\left[\exp\left(\frac{2U_0}{K}\right)-\exp\left(\frac{2U_0\zeta}{Ka_1}\right)\right]+\mu\frac{a_1+a_2}{(\zeta-a_1)(\zeta+a_2)}\left\{a_1\exp\left(\frac{2U_0\zeta}{Ka_1}\right)+a_2\exp\left(-\frac{2U_0\zeta}{Ka_2}\right)-(a_1+a_2)\right.$$
$$\left.\times\exp\left[\frac{2U_0}{K}\left(\frac{\zeta}{a_1}-\frac{\zeta}{a_2}-1\right)\right]-\left[\exp\left(\frac{2U_0\zeta}{Ka_1}\right)-\exp\left(-\frac{2U_0\zeta}{Ka_2}\right)\right]\zeta\right\}. \tag{28}$$

It follows that in the case that $\zeta(t)\equiv 0$ the probability flux is absent, i.e., the transport is impossible.

A. Transport of a Brownian particle under the action of small periodic force

In this item we consider the case when $\zeta(t)=B_0+B\sin\omega t$ with B_0, $B\ll a_{1,2}\,Ka_{1,2}/U_0$. Expanding $G(\zeta)$ in powers of ζ we obtain

$$G(\zeta)\approx\frac{U_0a_1a_2\zeta}{K^2(a_1+a_2)\sinh^2(U_0/K)}\left[1+\frac{a_2-a_1}{a_1a_2}\right.$$
$$\left.\times\left(\frac{U_0^2}{K^2\sinh^2(U_0/K)}+\frac{U_0}{K\tanh(U_0/K)}-2\right)\zeta(t)\right]$$
$$+\mu\frac{U_0a_1^2a_2^2\exp(-U_0/K)\zeta}{K^3(a_1+a_2)\sinh^3(U_0/K)}\left[1+\frac{a_2-a_1}{a_1a_2}\right.$$
$$\left.\times\left(\frac{2U_0^2}{K^2\sinh^2(U_0/K)}+\frac{U_0}{K\tanh(U_0/K)}-3\right)\zeta(t)\right]. \tag{29}$$

If $B=0$, $B_0\neq 0$, i.e., in addition to $f(x)$ a constant force acts upon the particle, then

$$\overline{\langle\dot{x}\rangle}\approx\left(\frac{U_0^2}{K^2\sinh^2(U_0/K_1)}+\mu\frac{U_0^2a_1a_2\exp(-U_0/K)}{K^3\sinh^3(U_0/K)}\right)B_0, \tag{30}$$

i.e., the particle moves in the direction of the constant force. (We note that in the absence of fluctuations, when $K\to 0$, $\overline{\langle\dot{x}\rangle}\to 0$, i.e., the transport of the particle is impossible.)

In another specific case, when $B_0=0$, $B\neq 0$, we obtain

$$\overline{\langle\dot{x}\rangle}\approx\left[\frac{U_0^2(a_2-a_1)}{K^2a_1a_2\sinh^2(U_0/K)}\left(\frac{U_0^2}{K^2\sinh^2(U_0/K)}\right.\right.$$
$$\left.+\frac{U_0}{K\tanh(U_0/K)}-2\right)+\mu\frac{U_0^2(a_2-a_1)\exp(-U_0/K)}{K^3\sinh^3(U_0/K)}$$
$$\left.\times\left(\frac{2U_0^2}{K^2\sinh^2(U_0/K)}+\frac{U_0}{K\tanh(U_0/K)}-3\right)\right]\frac{B^2}{2}. \tag{31}$$

It is easy to verify that in the absence of fluctuations the transport of the particle cannot occur, as for the case of a constant force.

As follows from Eq. (31), for $\mu=0$, $B_0=0$ the particle can move on average only in the direction of the slower rate of the potential change. However, for $\mu\neq 0$ and/or $B_0\neq 0$ the direction of motion can change, i.e., the flux reversal can occur. The dependencies of $2\overline{\langle\dot{x}\rangle}/B^2$ on U_0/K for $\mu=0$, $\mu a_1a_2/U_0=0.15$, $\mu a_1a_2/U_0=0.3$ are shown in Fig. 5 both for $B_0=0$ (a) and for $B_0a_1/B^2=-0.00125$ (b). We see that the flux reversal occurs at a moderately small value of U_0/K (U_0/K less than or of order 1).

It is evident that the flux reversal can be used for the separation of particles of different masses. Examples of the dependencies of $\overline{\langle\dot{x}\rangle}/B^2$ on $\mu a_1a_2/U_0$ for $U_0/K=0.6$, $U_0/K=0.7$, $U_0/K=0.8$ are illustrated in Fig. 6 for $B_0=0$ (a) and $B_0a_1/B^2=-0.00125$ (b). We see that in the presence of a constant force the separation of particles can be more effective. It should be noted that, as with noise-induced transport, the noise-induced separation of particles of different size or (and) mass is akin to vibrational separation [22].

The results obtained can be explained in the following manner. The noise-induced transport occurs if fluctuational transitions through the potential barrier are more frequent in one direction than in another. Because the probability of the transition through a certain potential barrier depends only on the height of this barrier and intensity of fluctuations, the

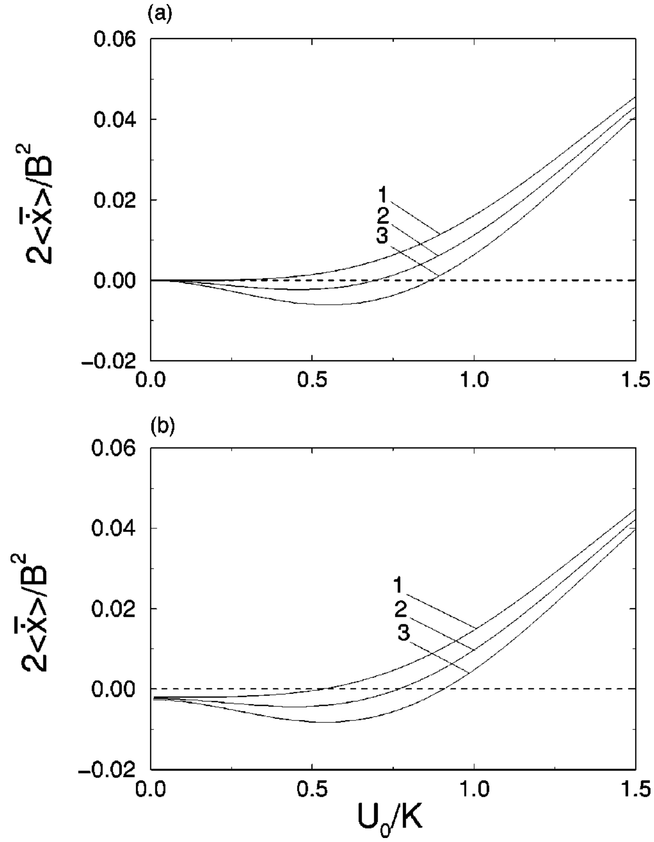

FIG. 5. The dependencies of $2\overline{\langle\dot{x}\rangle}/B^2$ on U_0/K for $L=1$, $a_1=1.25$, $a_2=5$, $\mu=0$ (curve 1), $\mu a_1 a_2/U_0=0.15$ (curve 2), $\mu a_1 a_2/U_0=0.3$ (curve 3), $B_0=0$ (a), and $B_0 a_1=-0.00125B^2$ (b).

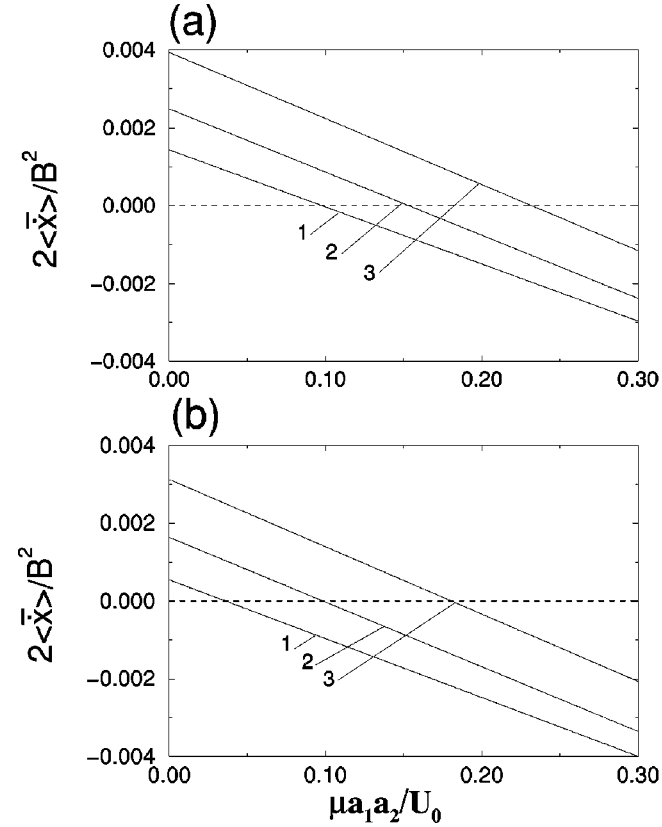

FIG. 6. The dependencies of $\overline{\langle\dot{x}\rangle}/B^2$ on $\mu a_1 a_2/U_0$ for $U_0/K=0.6$ (curve 1), $U_0/K=0.7$ (curve 2), and $U_0/K=0.8$ (curve 3), $B_0=0$ (a), and $B_0 a_1=-0.00125B^2$ (b).

transport is impossible in the absence of an additional force $\zeta(t)$. This force results in the fact that the probabilities of the transition through the potential barrier become different in different directions.

B. Transport of a Brownian particle under the action of a correlated random force

The problem of fluctuational transport of an overdamped Brownian particle in a viscous medium induced by thermal noise and a correlated random force that is a Markov process was studied by Doering, Horsthemke, and Riordan [20] and Bier [4]. However, concrete results were obtained by them only for dichotomous and so-called ''kangaroo'' processes.

A similar problem was also studied by Millonas and Dykman [17] but for the case where thermal noise is absent. In the approximation of sufficiently small correlation time of the random force they found nonzero current depending on the shape of the force spectral density. The inclusion of particle mass was performed by Bartussek, Hänggi, Lindner, and Schimansky-Geier [19], but the problem was tackled by numerically solving either initial stochastic equations or the corresponding Fokker-Planck equation.

First of all let us consider the case when the motion of a Brownian particle is described by Eq. (15), where the correlated random force $\zeta(t)$ is described by the equation

$$\ddot{\zeta}+\gamma\dot{\zeta}+\omega_0^2\zeta=\omega_0\gamma\xi_1(t). \tag{32}$$

We assume that $\xi_1(t)$ is white noise with zero mean value and intensity equal to K_1. We assume that $\xi_1(t)$ is noncorrelated with $\xi(t)$. It is easily shown that the spectral density and correlation function of the process $\zeta(t)$ are

$$S(\omega)=\frac{K_1\gamma^2\omega_0^2}{\pi[(\omega^2-\omega_0^2)^2+\gamma^2\omega^2]},$$

$$\langle\zeta(t)\zeta(t+\tau)\rangle=\frac{K_1\omega_0^2\gamma^2[\sqrt{4\omega_0^2-\gamma^2}\cos(\sqrt{\omega_0^2-\gamma^2/4}\;\tau)+\gamma\,\sin(\sqrt{\omega_0^2-\gamma^2/4}\;\tau)]}{2\omega_0^2\gamma\sqrt{4\omega_0^2-\gamma^2}}\exp\left(-\frac{\gamma\tau}{2}\right). \tag{33}$$

It follows that the variance of the random process $\zeta(t)$ is equal to $K_1\gamma/2$ and the correlation time is inverse to γ.

In the quasistationary approximation, which is valid for sufficiently large correlation times of the process $\zeta(t)$, we can use the approximate Fokker-Planck equation (19) and its solution (28) which should be averaged over ζ.

The stationary probability density of the variable ζ is equal to [24]

$$p(\zeta)=\sqrt{\frac{1}{\pi K_1\gamma}}\exp\left(-\frac{\zeta^2}{K_1\gamma}\right). \tag{34}$$

It coincides with the stationary probability density for a so-called Ornstein-Uhlenbeck process [25] described by the equation

$$\dot{\zeta}+\gamma\zeta=\gamma\xi_1(t). \tag{35}$$

It should be noted that Eqs. (35) and (32) are equivalent only with respect to the probability density, whereas they have radically different spectral densities and correlation functions: the spectral density and correlation function of the process $\zeta(t)$ described by Eq. (37) are

$$S(\omega)=\frac{K_1}{\pi(\omega^2\tau_c^2+1)},\quad \langle\zeta(t)\zeta(t+\tau)\rangle=\frac{K_1\gamma}{2}e^{-\gamma\tau}, \tag{36}$$

where $\tau_c=1/\gamma$ is the correlation time.

Taking into account Eq. (32) we find

$$\langle\dot{x}\rangle=\langle G(\zeta)\rangle L=L\sqrt{\frac{1}{\pi K_1\gamma}}\int_{-\infty}^{\infty}G(\zeta)\exp\left(-\frac{\zeta^2}{K_1\gamma}\right)d\zeta. \tag{37}$$

It follows from Eq. (34) that in the case of sufficiently small $K_1\gamma$ and sufficiently large K/U_0, when

$$K_1\gamma\ll a_{1,2}^2,\frac{K^2a_{1,2}^2}{U_0^2}, \tag{38}$$

the probability density $p(\zeta)$ declines rapidly with increasing $|\zeta|$. Therefore for calculating $\langle G(\zeta)\rangle$ we can expand $G(\zeta)$ as a power series in ζ. In this case we obtain the expression (29). Ignoring the linear term we find

$$G(\zeta)\approx\frac{U_0(a_2-a_1)}{K^2(a_1+a_2)\sinh^2(U_0/K)}\left[\frac{U_0^2}{K^2\sinh^2(U_0/K)}+\frac{U_0}{K\tanh(U_0/K)}-2+\mu\frac{a_1a_2\exp(-U_0/K)}{K\sinh(U_0/K)}\times\left(\frac{2U_0^2}{K^2\sinh^2(U_0/K)}+\frac{U_0}{K\tanh(U_0/K)}-3\right)\right]\zeta^2(t). \tag{39}$$

Substituting Eq. (39) into Eq. (37) we have

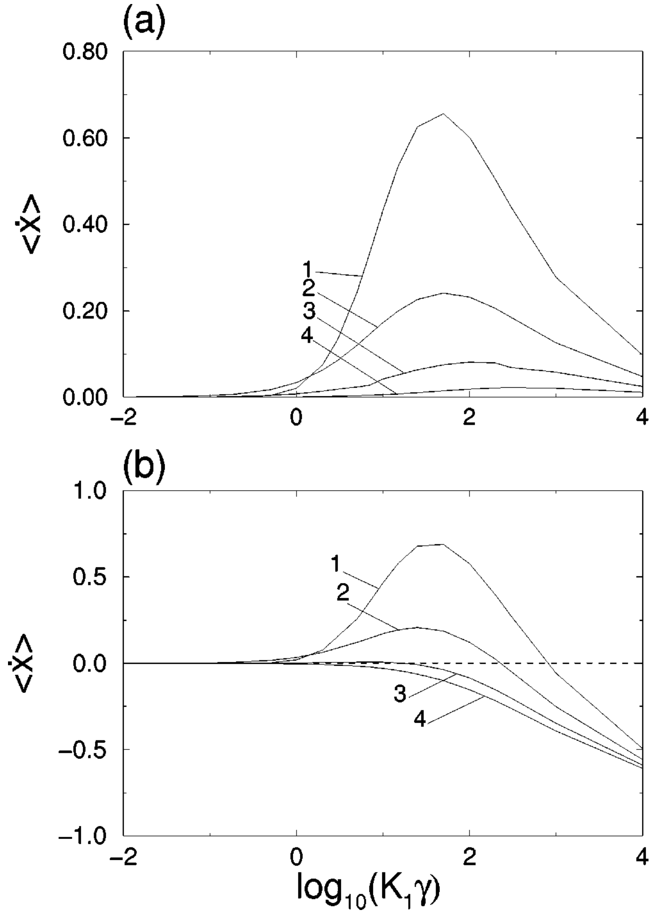

FIG. 7. The dependencies of $\langle\dot{x}\rangle$ on $\log_{10}(K_1\gamma)$ for $a_1=1.25$, $a_2=5$, $K/U_0=0.1$ (curve 1), $K/U_0=0.5$ (curve 2), $K/U_0=1$ (curve 3), $K/U_0=2$ (curve 4), $\mu_0=0$ (a), and $\mu a_1a_2/U_0=0.25$ (b).

$$\langle\dot{x}\rangle\approx\frac{U_0^2(a_2-a_1)}{K^2a_1a_2\sinh^2(U_0/K)}\left[\frac{U_0^2}{K^2\sinh^2(U_0/K)}+\frac{U_0}{K\tanh(U_0/K)}-2+\mu\frac{a_1a_2\exp(-U_0/K)}{K\sinh(U_0/K)}\times\left(\frac{2U_0^2}{K^2\sinh^2(U_0/K)}+\frac{U_0}{K\tanh(U_0/K)}-3\right)\right]\frac{K_1\gamma}{2}. \tag{40}$$

The dependencies of $\langle\dot{x}\rangle$ on $K_1\gamma$ calculated numerically from the expression (37), in view of Eq. (28), for $a_1=1.25$, $a_2=5$ and different values of K/U_0 are shown in Fig. 7 for $\mu=0$ (a) and $\mu a_1a_2/U_0=0.25$ (b). It is seen from this figure that for a fixed value of K/U_0 and $\mu=0$ the value of $\langle\dot{x}\rangle$ first increases as $K_1\gamma$ increases and then slowly decreases approaching zero as $K_1\gamma\to\infty$. The peak of $\langle\dot{x}\rangle$ is located at greater values of $K_1\gamma$, the greater K/U_0 is. The inclusion of particle mass results in a shift of the peak in the direction of smaller values of $K_1\gamma$ which is the larger the larger is K/U_0. From a certain value of K/U_0 onwards the values of $\langle\dot{x}\rangle$ become negative for any $K_1\gamma$ and the dependence of $\langle\dot{x}\rangle$ on $K_1\gamma$ becomes monotonically descending.

The dependencies of $\langle\dot{x}\rangle$ on K/U_0 for a fixed value of $K_1\gamma$, shown in Fig. 8, are of a somewhat different form.

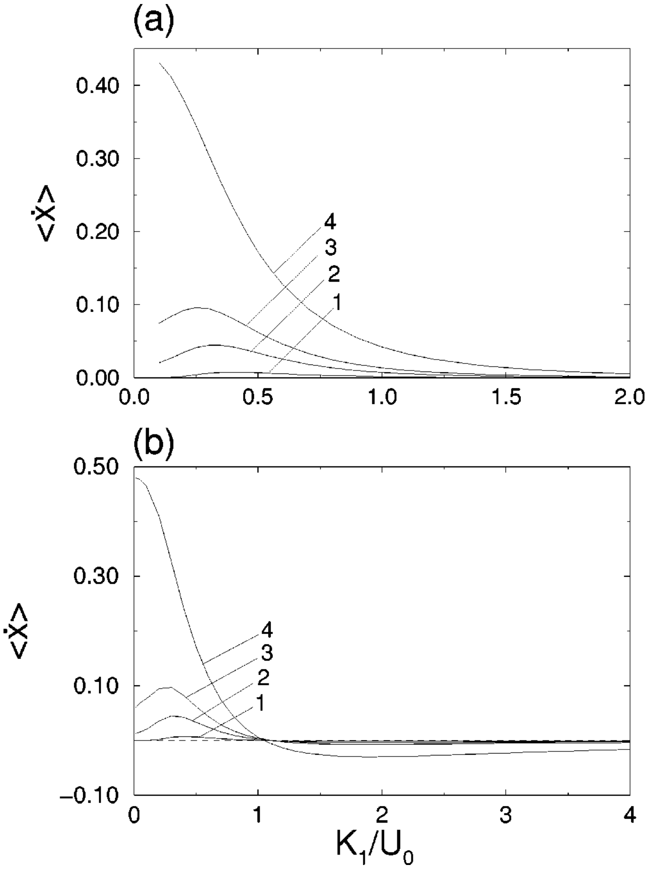

FIG. 8. The dependencies of $\langle \dot{x} \rangle$ on K/U_0 for $a_1=1.25$, $a_2=5$, $K_1\gamma=0.2$ (curve 1), $K_1\gamma=1$ (curve 2), $K_1\gamma=2$ (curve 3), $K_1\gamma=10$ (curve 4), $\mu_0=0$ (a), and $\mu a_1 a_2/U_0=0.25$ (b).

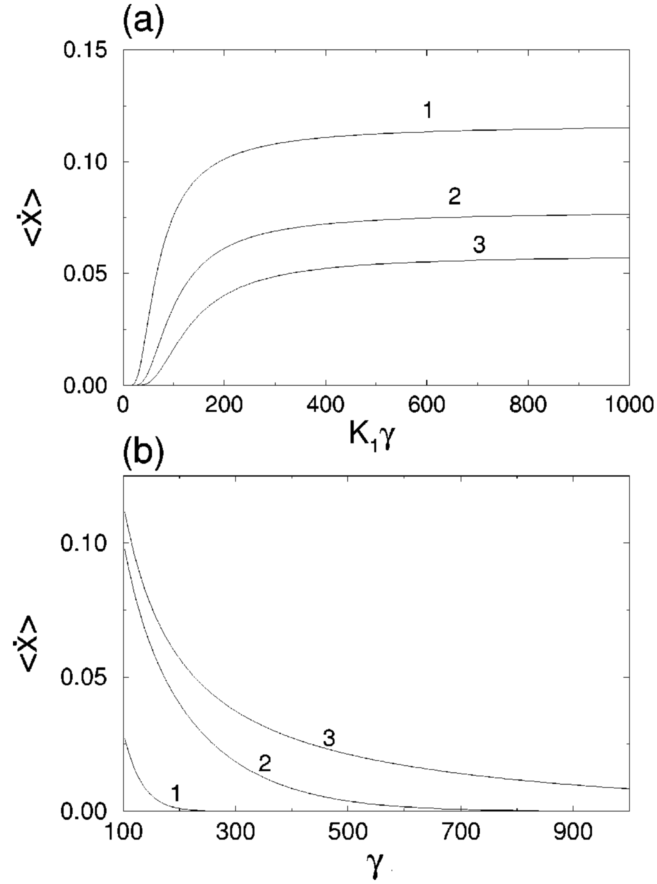

FIG. 9. (a) The dependencies of $\langle \dot{x} \rangle$ on $K_1\gamma$ for $a_1=1.25$, $a_2=5$, $U_0=1$, $\gamma=100$ (curve 1), $\gamma=150$ (curve 2), and $\gamma=200$ (curve 3). (b) The dependencies of $\langle \dot{x} \rangle$ (γ for 1), $K_\gamma=200_1\gamma=1000$ (curve 3).

Even for $\mu=0$ (a) they have a maximum at a certain value of $K/U_0\neq 0$ only for $K_1\gamma$ nonexceeding a certain value, whereas for greater $K_1\gamma$ the dependencies become monotonically descending. The inclusion of particle mass [see Fig. 8(b)] has little or no effect on the character of these dependencies for K/U_0 less than or of order 1. For greater values of K/U_0 the values of $\langle x \rangle$ become negative and then, for $K/U_0\to\infty$, they also tend to zero.

The problem considered can also be solved approximately in the other limiting case when the correlation time of the random process $\zeta(t)$ is sufficiently small. This is precisely the case which is mainly considered in Ref. [17] but using an alternative way. The solution of this problem is the simplest if the thermal noise $\xi(t)$ is absent, the random force $\zeta(t)$ is described by Eq. (35), and $\mu=0$. In this case we can obtain from Eqs. (15),(35) the following differential equation of second order:

$$\tau_c\ddot{x}+[1+\tau_c f'(x)]\dot{x}+f(x)=\xi_1(t), \tag{41}$$

where $\tau_c=1/\gamma$ is the correlation time playing the role of a small parameter.

The approximate equation for the probability density $w(x)$ corresponding to Eq. (41) is (see Ref. [24])

$$\frac{\partial w}{\partial t}=\frac{\partial}{\partial x}\left(f(x)w+\frac{K_1}{2}\frac{\partial}{\partial x}\{[1-\tau_c f'(x)]w\}\right). \tag{42}$$

In the stationary case we obtain from here

$$-\left(f(x)-\tau_c\frac{K_1}{2}f''(x)\right)w-\frac{K_1}{2}[1-\tau_c f'(x)]\frac{dw}{dx}=G, \tag{43}$$

where G is the probability flux.

A solution of Eq. (43) with a precision of terms of order τ_c is

$$w(x)=\exp\left[-\frac{2}{K_1}\left(\int_0^x f(x')dx'+\frac{\tau_c}{2}f^2(x)\right)+\tau_c f'(x)\right]\times\left\{C-\frac{2G}{K_1}\int_0^x\exp\left[\frac{2}{K_1}\left(\int_0^{x'}f(y)dy+\frac{\tau_c}{2}f^2(x')\right)dx'\right]\right\}. \tag{44}$$

Using the periodicity and normalization conditions and assuming that the function $f(x)$ is close to that is described by Eq. (16) we find

$$\langle\dot{x}\rangle = a_1 a_2 U_0\left[\exp\left(-\frac{a_1^2}{K_1\gamma}\right)-\exp\left(-\frac{a_2^2}{K_1\gamma}\right)\right] \times\left\{2K_1\left[a_2\exp\left(-\frac{a_1^2}{K_1\gamma}\right)+a_1\exp\left(-\frac{a_2^2}{K_1\gamma}\right)\right] \times\sinh^2\frac{U_0}{K_1}+(a_2-a_1)\left[K_1\exp\left(-\frac{U_0}{K_1}\right)\sinh\frac{U_0}{K_1}-U_0\right] \times\left[\exp\left(-\frac{a_1^2}{K_1\gamma}\right)-\exp\left(-\frac{a_2^2}{K_1\gamma}\right)\right]\right\}^{-1}. \quad (45)$$

For a fixed γ the value of $\langle\dot{x}\rangle$ first increases with increasing $K_1\gamma$ and then remains nearly constant [see Fig. 9(a)]. For a fixed $K_1\gamma$ the value of $\langle\dot{x}\rangle$ decreases monotonically as γ increases [Fig. 9(b)].

We note that the expression (45) and the dependencies found by us differ qualitatively from the corresponding results obtained in Ref. [17].

In conclusion it should be noted that, as easily seen from Eq. (45), $\langle\dot{x}\rangle\to 0$ for $\tau_c\to 0$, but it is nonzero for any finite τ_c. Undeniably, this result is in contradiction with the second law of thermodynamics because thermal equilibrium noise is not white but colored with small correlation time. It testifies that the ratchet model considered should be corrected much as was done by Stratonovich for the diode model.

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